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Differential representation of multipole fields

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Abstract
It is shown that the potential of an electrostatic or magnetostatic 2^l-pole can be expressed as the composition of l directional derivatives of the function $1/r$ along l directions, not necessarily distinct.

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1. Introduction

The 2^l-pole moment of an electric charge distribution is represented by a tracefree totally symmetric l-index tensor which has $2l + 1$ independent components (see, e.g., [1–5]). (A tensor is tracefree if all its traces are equal to zero, see equation (5).) In particular, the dipole moment is represented by a vector, $\mathbf{p}$, and its contribution to the electrostatic potential is

$$\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = -\frac{1}{4\pi\varepsilon_0}\mathbf{p}\cdot\nabla\frac{1}{r}. \tag{1}$$

Noting that $\mathbf{p}\cdot\nabla(1/r)$ is the directional derivative of $1/r$ along the direction of $\mathbf{p}$ and writing $\mathbf{p} = q\mathbf{a}$, where q is some positive quantity with units of electric charge and $\mathbf{a} = \mathbf{p}/q$ is a vector parallel to $\mathbf{p}$, we have

$$-\mathbf{p}\cdot\nabla\frac{1}{r} = q(-\mathbf{a})\cdot\nabla\frac{1}{r} = q\lim_{s\to 0}\frac{1}{s}\left[\frac{1}{|\mathbf{r}-s\mathbf{a}|} - \frac{1}{|\mathbf{r}|}\right] = \lim_{s\to 0}\left[\frac{q/s}{|\mathbf{r}-s\mathbf{a}|} - \frac{q/s}{|\mathbf{r}|}\right]$$

which corresponds to the well-known fact that the dipole field (1) is equal to the limit as s goes to zero of the field produced by a point charge $-q/s$ placed at the origin and a point charge q/s at the point $s\mathbf{a}$.

Similarly, it can be shown that the directional derivative of the dipole field (1) along any direction is exactly a quadrupole field and, more generally, any directional derivative of a 2^l-pole field is a 2^{l+1}-pole field. The aim of this paper is to show that for an arbitrary bounded electric charge or current distribution and for any value of l ($l = 1, 2, 3, \ldots$), there exist l

vectors, $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$, not necessarily distinct, such that the 2^l-pole term of the electrostatic or magnetostatic potential is given by

$$(-1)^l (\mathbf{a} \cdot \nabla)(\mathbf{b} \cdot \nabla) \cdots (\mathbf{f} \cdot \nabla) \frac{1}{r}. \tag{2}$$

The vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ can be chosen in such a way that they all have the same magnitude and, therefore, their common magnitude (one real number) and the two variables specifying the direction of each of the l vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$, give $2l + 1$ independent real numbers that determine the 2^l-pole term of the potential. According to equation (2), the quadrupole term, for instance, is equivalent to the limit as s goes to zero of the field of two dipoles, one with dipole moment $-\mathbf{a}/s$ at the origin and another with dipole moment $\mathbf{a}/s$ at the point $s\mathbf{b}$.

Expression (2) follows from the fact that any tracefree totally symmetric l-index tensor can be expressed as the tracefree part of the symmetrized tensor product of l vectors [6, 7]. An elementary proof of this result, for the case where $l = 2$, is given below.

The usefulness of expression (2) comes from the fact that, instead of making use of Cartesian tensors or spherical harmonics, it only involves ordinary vectors and provides a simple way of viewing any multipole moment of an arbitrary charge or current distribution as a set of vectors.

In section 2 the multipole expansion of the electrostatic field is considered and in section 3 an analogous treatment for the magnetostatic field is given, where we also present a simple derivation of the expression for the multipole moments of a current distribution.

2. Multipole expansion of the electrostatic field

By expanding the function $|\mathbf{r} - \mathbf{r}'|^{-1}$ in a power series, one finds that the (external) potential of a bounded static electric charge distribution is given by

$$\begin{aligned}\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \Bigg[\frac{1}{r} \int \rho(\mathbf{r}')\,\mathrm{d}v' + \frac{x_i}{r^3} \int \rho(\mathbf{r}') x_i'\,\mathrm{d}v' + \frac{x_i x_j}{2r^5} \int \rho(\mathbf{r}')(3x_i' x_j' - r'^2 \delta_{ij})\,\mathrm{d}v' \\ + \frac{x_i x_j x_k}{2r^7} \int \rho(\mathbf{r}')(5x_i' x_j' x_k' - r'^2 x_i' \delta_{jk} - r'^2 x_j' \delta_{ki} - r'^2 x_k' \delta_{ij})\,\mathrm{d}v' + \cdots \Bigg]\end{aligned} \tag{3}$$

where x_i and x_i' are the Cartesian components of $\mathbf{r}$ and $\mathbf{r}'$, respectively, $r = |\mathbf{r}|$, $r' = |\mathbf{r}'|$, ρ is the electric charge density. Throughout this paper each repeated index in a product $i, j, k, \ldots$ implies a summation over 1, 2, 3. The integrals in equation (3) and in the expressions below are over all the space or, since the charge or current distribution is bounded, over the region containing the sources of the field. The multipole expansion (3) is of the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{2l+1}} \underbrace{x_i x_j \cdots x_p}_{l \text{ factors}} M^{(l)}_{ij\cdots p} \tag{4}$$

where $M^{(l)}_{ij\cdots p}$ is a tracefree totally symmetric l-index tensor, i.e., $M^{(l)}_{ij\cdots r\cdots s\cdots p} = M^{(l)}_{ij\cdots s\cdots r\cdots p}$ and

$$M^{(l)}_{ij\cdots s\cdots s\cdots p} = 0 \qquad (l \geqslant 2). \tag{5}$$

By comparing with equation (3) one finds that the lowest multipole moments are given by

$$\begin{aligned} M^{(0)} &= \int \rho(\mathbf{r}')\,\mathrm{d}v' \qquad M^{(1)}_i = \int \rho(\mathbf{r}') x_i'\,\mathrm{d}v' \\ M^{(2)}_{ij} &= \frac{1}{2} \int \rho(\mathbf{r}')(3x_i' x_j' - r'^2 \delta_{ij})\,\mathrm{d}v' \\ M^{(3)}_{ijk} &= \frac{1}{2} \int \rho(\mathbf{r}')(5x_i' x_j' x_k' - r'^2 x_i' \delta_{jk} - r'^2 x_j' \delta_{ki} - r'^2 x_k' \delta_{ij})\,\mathrm{d}v'. \end{aligned}$$

The components $M^{(1)}_i$ and $2M^{(2)}_{ij}$ are usually denoted as p_i and Q_{ij}, respectively [1–3].

On the other hand, one finds that, for $r \neq 0$,

$$\frac{\partial}{\partial x_i}\frac{1}{r} = -\frac{x_i}{r^3} \qquad \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\frac{1}{r} = \frac{3x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}$$

$$\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\frac{1}{r} = -\frac{15x_i x_j x_k}{r^7} + \frac{3x_k\delta_{ij}}{r^5} + \frac{3x_j\delta_{ik}}{r^5} + \frac{3x_i\delta_{jk}}{r^5}, \ldots$$

therefore, making use of (5), equation (4) can also be expressed as

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l-1)!!}M^{(l)}_{ij\cdots p}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_p}\frac{1}{r} \tag{6}$$

(with $(-1)!! \equiv 1$). It may be noted that [4]

$$M^{(l)}_{ij\cdots p} = \frac{(-1)^l}{l!}\int \rho(\mathbf{r}')r'^{2l+1}\frac{\partial}{\partial x'_i}\frac{\partial}{\partial x'_j}\cdots\frac{\partial}{\partial x'_p}\frac{1}{r'}\,\mathrm{d}v'.$$

Even though, for $l > 1$, not every l-index tensor is equal to the tensor product of l vectors, it turns out that every tracefree totally symmetric l-index tensor is equal to the tracefree part of the symmetrized tensor product of l vectors [6, 7]. For instance, if Q_{ij} is a tracefree symmetric 2-index tensor, then there exist two vectors, v_i, w_i, such that

$$Q_{ij} = \tfrac{1}{2}(v_i w_j + v_j w_i) - \tfrac{1}{3}(\mathbf{v}\cdot\mathbf{w})\delta_{ij}.$$

Indeed, as is well known, if (Q_{ij}) is a symmetric 3×3 real matrix then there exists an orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, formed by normalized eigenvectors of (Q_{ij}), such that

$$Q_{ij} = \lambda a_i a_j + \mu b_i b_j + \nu c_i c_j \tag{7}$$

where λ, μ and ν are the corresponding eigenvalues, which are all real. If the trace of (Q_{ij}) is equal to zero, then $\lambda + \mu + \nu = 0$. Furthermore, the condition that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis is equivalent to $a_i a_j + b_i b_j + c_i c_j = \delta_{ij}$, thus, from (7), we have

$$Q_{ij} = (2\lambda + \mu)a_i a_j + (2\mu + \lambda)b_i b_j - (\lambda + \mu)\delta_{ij}. \tag{8}$$

If now we assume that λ and μ are the greatest and the smallest eigenvalues of (Q_{ij}), respectively, then $2\lambda + \mu$ and $(-2\mu - \lambda)$ are greater than or equal to zero. Letting

$$\mathbf{v} \equiv \sqrt{2\lambda + \mu}\,\mathbf{a} + \sqrt{-2\mu - \lambda}\,\mathbf{b} \qquad \mathbf{w} \equiv \sqrt{2\lambda + \mu}\,\mathbf{a} - \sqrt{-2\mu - \lambda}\,\mathbf{b} \tag{9}$$

from (8) one obtains

$$Q_{ij} = \tfrac{1}{2}(v_i w_j + v_j w_i) - \tfrac{1}{3}(\mathbf{v}\cdot\mathbf{w})\delta_{ij} \tag{10}$$

as stated above. Since $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis, it follows from (9) that $|\mathbf{v}|^2 = |\mathbf{w}|^2 = \lambda - \mu$. Equation (10) means that any tracefree symmetric tensor Q_{ij} is the tracefree part of the symmetrized tensor product of two vectors of the same length.

From equations (9) we see that $\mathbf{v}$ and $\mathbf{w}$ are parallel to each other (i.e., $\mathbf{v} = \pm\mathbf{w}$) if and only if $2\mu + \lambda = 0$ or $2\lambda + \mu = 0$. Recalling that $\lambda + \mu + \nu = 0$, this means that $\nu = \mu$ or $\nu = \lambda$, respectively. Thus, $\mathbf{v}$ and $\mathbf{w}$ are parallel to each other if and only if two of the eigenvalues of (Q_{ij}) coincide.

It turns out that a result similar to (10) holds for tracefree symmetric tensors with any number of indices. If $t_{ij\ldots k}$ is a totally symmetric tracefree l-index tensor, then there exist l vectors of the same length, $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$, such that $t_{ij\ldots k}$ is the tracefree part of the symmetrized tensor product of $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$. The proof of this fact in general, which at the same time provides a method to find the vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$, is given in [6, 7] making use of the two-component spinor formalism.

Another characterization of the vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ is that if the z-axis coincides with one of the vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$, then the spherical multipole moment [1, 2] q_{ll} vanishes [5] (in the case of the quadrupole moment, Q_{ij}, this amounts to $Q_{11} - Q_{22} - 2\mathrm{i}Q_{12} = 0$).

Among other things, this means that, in the same way as the dipole moment can be represented by a vector, the quadrupole moment can be represented by two vectors of the same length (not necessarily distinct), the octopole moment can be represented by three vectors of the same length, and so on.

According to the foregoing results, the quadrupole moment $M^{(2)}_{ij}$ can be expressed in the form $M^{(2)}_{ij} = \frac{1}{2}(v_i w_j + v_j w_i) - \frac{1}{3}(\mathbf{v}\cdot\mathbf{w})\delta_{ij}$, for some vectors v_i and w_i; therefore, apart from the factor $1/(4\pi\varepsilon_0)$, the quadupole term in equation (6) can be written as

$$\frac{1}{3}\left[\frac{1}{2}(v_i w_j + v_j w_i) - \frac{1}{3}(\mathbf{v}\cdot\mathbf{w})\delta_{ij}\right]\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{r} = \frac{1}{3}v_i\frac{\partial}{\partial x_i}w_j\frac{\partial}{\partial x_j}\frac{1}{r} = \frac{1}{3}(\mathbf{v}\cdot\nabla)(\mathbf{w}\cdot\nabla)\frac{1}{r}$$

since $\nabla^2(1/r) = 0$, for $r \neq 0$. In a similar manner, the field of a 2^l-pole is of the form

$$\frac{(-1)^l}{4\pi\varepsilon_0}(\mathbf{a}\cdot\nabla)(\mathbf{b}\cdot\nabla)\cdots(\mathbf{f}\cdot\nabla)\frac{1}{r}$$

for some vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$, which can be chosen in such a way that they all have the same magnitude.

3. Multipole expansion of the magnetostatic field

The magnetic field produced by a bounded stationary electric current distribution also has a multipole expansion and, for $l > 1$, the magnetic field produced by a magnetic 2^l-pole moment is of the same form as the electric field produced by an electric 2^l-pole moment [1–5]. At a point outside an sphere centred at the origin containing the current distribution, the magnetic field can be expressed in the form $\mathbf{B} = -\nabla\phi_{\mathrm{M}}$ and the magnetic scalar potential, ϕ_{M}, has a multipole expansion similar to that given by equations (4) or (6) (with $M^{(0)} = 0$), namely

$$\begin{aligned}\phi_{\mathrm{M}}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\frac{1}{r^{2l+1}}\underbrace{x_i x_j\cdots x_p}_{l\ \text{factors}} M^{(l)}_{ij\cdots p} \\ &= \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\frac{(-1)^l}{(2l-1)!!}M^{(l)}_{ij\cdots p}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_p}\frac{1}{r}\end{aligned} \tag{11}$$

where now $M^{(l)}_{ij\cdots p}$ is a tracefree totally symmetric l-index tensor given by an integral containing the electric current density, $\mathbf{J}$. For instance

$$M^{(1)}_i = \frac{1}{2}\int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))_i\,\mathrm{d}v' \qquad M^{(2)}_{ij} = \int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))_{(i}x'_{j)}\,\mathrm{d}v' \tag{12}$$

where the parentheses denote symmetrization on the indices enclosed (e.g., $t_{(ij)} = \frac{1}{2}(t_{ij} + t_{ji})$). In general, $M^{(l)}_{ij\cdots p}$ is the tracefree part of the symmetric tensor [4, 5]

$$\frac{(2l-1)!!}{(l-1)!(l+1)}\int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))_{(i}x'_j\cdots x'_{p)}\,\mathrm{d}v'. \tag{13}$$

Indeed, starting from the elementary expression [1–3]

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}v'$$

for the field produced by the current density $\mathbf{J}$, noting that $(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')=0$, we have

$$\mathbf{r}\cdot\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{r}'\cdot\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}v'=\frac{\mu_0}{4\pi}\int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}v'.$$

Hence, using again the expansion of $|\mathbf{r}-\mathbf{r}'|^{-1}$ and writing $\mathbf{B}=-\nabla\phi_{\mathrm{M}}$, it follows that

$$-r\frac{\partial\phi_{\mathrm{M}}}{\partial r}=\frac{\mu_0}{4\pi}\int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))\cdot\nabla'\left[\frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}+\frac{3(\mathbf{r}\cdot\mathbf{r}')^2-r^2r'^2}{2r^5}+\cdots\right]\mathrm{d}v'$$

which leads to the expression

$$\phi_{\mathrm{M}}=\frac{\mu_0}{4\pi}\left[\frac{x_i}{2r^3}\int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))_i\,\mathrm{d}v'+\frac{x_ix_j}{r^5}\int(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))_ix'_j\,\mathrm{d}v'+\cdots\right]$$

(cf (3)). Since this last expression is a solution of the Laplace equation, it gives, up to a constant term, the desired scalar potential. Then, comparison with (11) yields equations (12).

As in the case of the multipole expansion of the electrostatic field, the fact that the Cartesian multipole moments $M^{(l)}_{ij\cdots p}$ appearing in equation (11) are totally symmetric and tracefree, implies that the field of a magnetic 2^l-pole is of the form

$$\frac{\mu_0}{4\pi}(-1)^l(\mathbf{a}\cdot\nabla)(\mathbf{b}\cdot\nabla)\cdots(\mathbf{f}\cdot\nabla)\frac{1}{r}$$

where $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ are l vectors of the same magnitude.

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